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Translated by A.Y.

ON A PLASTIC SHEAR WAVE

PMM Vol. 32, No. 3, 1968, pp. 502-504

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(Received December 21, 1967)

A self-similar solution of the problem of propagation of a perturbation produced by a glancing collision against the boundary of a half-space whose material conforms to the Prandtl-Reuss equations is constructed.

Simple conditions of solvability of the problem for two types of boundary conditions are constructed. These boundary conditions correspond to the cases of 1) total adhesion and 2) Coulomb dry friction.

1. The Prandtl-Reuss equations are sometimes used to describe the motion of a soil under large loads [1]. Problems of this type usually contain two space variables and time, and can only be solved numerically. In some such problems it is necessary to consider the interaction of waves with a hard surface. The boundary conditions which this requires have not been investigated sufficiently.

It is natural to attempt to gain insight into the situation by way of some simple problem. We shall consider an elementary case which nevertheless retains some of the salient features of complex problems of wave and surface interaction.

Let a hard slab be pressed by the force σ_0 against the boundary of a half-space. At $t = 0$ the slab is set in motion with the constant velocity v_0 directed along the boundary.

For $t < 0$ the half-space is a rest, and the stress it experiences is constant.

Since the basic equations allow for the appearance of tangent stresses in the medium, we can stipulate at the boundary either an adhesion condition or the dry friction law natural in solid body contact.

In Section 2 we shall show that under the adhesion condition the problem has a solution only for velocities restricted by the inequality $v_0 \leq v_*$; a unique solution does not exist for $v_0 > v_*$. It will be shown that a solution exists only if the coefficient satisfies some (quite simple) inequality.

The notation is as follows: x is a coordinate (the x-axis is directed into the half-space); u is the velocity along x; v is the velocity along the normal to x; K is the bulk modulus; G is the shear modulus; θ is the volume compression; σ is the stress along x; τ is the tangent stress; p is the hydrostatic pressure; f is the coefficient of friction. The plasticity condition is

$$^3/_4\,(\sigma + p)^2 + \tau^2 = T^2, \quad T = {}^1/_2\,\sqrt{3}\,kp \tag{1.1}$$

We assume that the density is equal to unity, and that

$$K = K_1 \ \text{for}\ \theta^{\cdot} > 0 \qquad K = K_2 \ \text{for}\ \theta^{\cdot} < 0 \qquad ((K_1,\ K_2 = \text{const})$$
$$K > G, \qquad kK_2 > {}^4/_3 G > kK_1 \qquad (G,\ k = \text{const}) \tag{1.2}$$

We also introduce the ancillary variables

$$s = +\sqrt{1 - \tau^2 / T^2}, \qquad a = x / t$$

The next step is to solve the system consisting of the equations of motion, the continuity equation, and the Prandtl-Riesz equations,

$$\frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial x}, \quad \frac{\partial v}{\partial t} = \frac{\partial \tau}{\partial x}, \quad \frac{\partial p}{\partial t} = -K\frac{\partial u}{\partial x}$$
$$\frac{\partial \tau}{\partial t} + \left[G\frac{\sigma + p}{T^2}\frac{\partial u}{\partial x} + G\frac{\tau}{T^2}\frac{\partial v}{\partial x} - \frac{\sqrt{3}}{2}k\frac{\partial p}{\partial t}\right]\tau = G\frac{\partial v}{\partial x}$$

under the initial and boundary conditions

$$\tau = 0, \quad u = 0, \quad v = 0, \quad \sigma = \sigma_0 = -kp_0 \quad \text{for}\ t = 0$$
$$u = 0, \quad v = v_0 \quad \text{for}\ t > 0 \quad \text{for}\ x = 0 \quad \text{for}\ (\text{п. } 3)$$
$$u = 0, \quad \tau = f\,|\,\sigma\,|\,\text{sgn}\,[v_0 - v\,(t, +0)] \quad \text{for}\ t > 0,\ x = 0$$

We now assume that $v_0 > 0$ (the case $v_0 < 0$ results when we change sign).

From [2] we infer that the assumptions just fomulated admit of the following solutions of the basic system which depend only on a : 1) a constant solution; 2) a strong discontinuity which propagates at the velocity $\sqrt{(1+k)K}$, 3) a simple centered wave. The following relations are fulfilled in the domain occupied by the wave:

$$a = \sqrt{\omega - \sqrt{\omega^2 - GKs\,(s+k)}} \tag{1.3}$$

$$p = p_1\varphi(s), \qquad \sigma = -(1 + ks)\,p, \qquad \tau = {}^1/_2\,\sqrt{3}\,\sqrt{1 - s^2}\,kp \tag{1.4}$$

$$u = -\int_1^s \frac{d\sigma}{ds}\frac{ds}{a} + u_1, \qquad v = +\int_1^s \frac{d\tau}{ds}\frac{ds}{a} + v_1 \tag{1.5}$$

$$\omega = {}^1/_2\,[(1 + ks)\,K - {}^1/_3\,(4 - s^2)\,G] \qquad (p_1 = p_0 + \Delta p)$$

$$\varphi(s) = \exp\int_1^s \frac{kKd\xi}{a^2(\xi) - (1 + ks)\,K}$$

Here Δp is the intensity of the strong discontinuity. Formulas (1.3)-(1.5) define parametrically the dependences of σ, p, τ, u, and v on a.

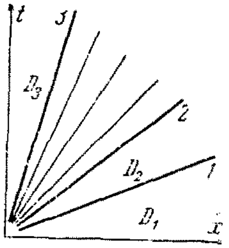

Fig. 1

Straight line *1* in the plane xt (see Fig. 1) represents the strong discontinuity. Straight lines *2* and *3* represent the forward and rear fronts of the simple wave. Straight line *2* is described by Eq. $x = \sqrt{G}t$, $s = 1$; straight lines *3* is described by $x = a\,(s_2)t$, where s_2 is an arbitrary constant satisfying the inequality $0 \leqslant s_2 \leqslant 1$. The solution is constant in the domains D_1, D_2, D_3. Quantities associated with the domain D_i will be denoted by the subscript i.

The boundary conditions are satisfied by means of the two arbitrary constant Δp and s_2.

2. Let us first consider the adhesion condition, i.e. the case

where $u_2 = 0$, $v_2 = v_0$. To begin with, we note that $v_1 = 0$, and that $s_2 \neq 1$ if $v_0 > 0$. From (1.4) and (1.5) we conclude that $u_2 > u_1$ and $u_1 < 0$, $\Delta p < 0$, i.e. that the strong discontinuity is introduced in order to compensate the change in u at the simple wave. Making use of the boundary conditions, the relations at the discontinuity, and expressions (1.4) and (1.5), we obtain two equations for p_1 and s_2,

$$p_2(1 + h(s_2)) - p_0 = 0 \tag{2.1}$$

$$p_1\psi(s_2) - v_0 = 0 \tag{2.2}$$

where

$$h(s_2) = \int_1^{s_2} \frac{k\sqrt{(1+k)K}\,\varphi(s)\,a(s)\,ds}{a^2(s) - (1+ks)K} \tag{2.3}$$

$$\psi(s_2) = \int_1^{s_2} k\frac{\sqrt{3}}{2}\frac{[(k+s)K - sa^2(s)]}{a^2(s) - (1+ks)K}\frac{\varphi(s)\,ds}{a\sqrt{1-s^2}} \tag{2.4}$$

Equations (1.3) and (1.2) imply that $a^2 < G$ and $a = O(\sqrt{s})$ as $s \to 0$, $k \neq 0$. Simple analysis shows that the functions $h(s_2)$ and $\psi(s_2)$ are positive and bounded, and that $h'(s_2) < 0$ $\psi'(s_2) < 0$ and $d\psi(1+h)^{-1}/ds_2 < 0$.

From (2.2) and (2.1) we find that

$$\psi(s_2)[1 + h(s_2)]^{-1} = v_0 \tag{2.5}$$

Since the left side of (2.5) is a monotonous and bounded function of s_2, we can determine s_2 from (2.5) if

$$v_0 \leqslant v_* \equiv \psi(0)/[1 + h(0)]$$

Assuming that the coefficient of static friction f_0 is given by the formula $f_0 = \max|\tau_2/\sigma_2|$, we find from (1.4) $f_0 = k\sqrt{3}/2$.

Thus, stipulation of adhesion conditions at the boundary enables us to solve the problem only if $v_0 \leq v_*$.

3. Now let the conditions at $x = 0$ be those of dry friction, i.e. $u_2 = 0$. $\tau_2 = f|\sigma_2|\,\mathrm{sgn}(v_0 - v_2)$. It follows from (1.5) that $\mathrm{sgn}\,\tau_2 = \mathrm{sgn}\,v_2$. Since $v_0 > 0$, the boundary conditions imply that $\tau_2 > 0$, $v_2 < v_0$.

The condition $u_2 = 0$ once again yields Eq. (2.1). The condition for τ together with (1.4) yields the second equation

$${}^1/_2 k\sqrt{3}\sqrt{1 + s_2{}^2} = f(1 + ks_2) \tag{3.1}$$

It is easy to determine s_2 and p_1 from (3.1) and (2.1), and then to construct the solution of the problem in explicit form.

Since $0 \leq s_2 < 1$, it follows that (3.1) is solvable only if $f \leq k\sqrt{3}/2 = f_0$. In other words, the problem is solvable only if the coefficient of sliding friction is smaller than or equal to the coefficient of static friction.

We must also verify the condition $v_2 < v_0$. This condition can be rewritten as

$$v_0 > \psi(s_2)/[1 + h(s_2)] \equiv v_- \tag{3.2}$$

where s_2 is given by (3.1) and where ψ and h are defined by Formulas (2.4) and (2.3).

The conditions of solvability therefore become

$$v_0 > v_-, \quad f \leqslant f_0$$

Since (3.2) implies that $v_- < v_*$, it follows that the problem can be solved for any $v > 0$, provided that the adhesion condition is used for $v_0 \leq v_-$ and the dry friction condition with $f < f_0$ is used for $v_0 > v_-$. This clearly does not exhaust the problem of boundary conditions in the general case, since v_- is not known in advance.

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Translated by A. Y.

ON THE STABILITY OF STEADYSTATE MOTIONS

PMM Vol. 32, No. 3, 1968, pp. 504-508

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(Received January 23, 1968)

Paper [1] describes a method for investigating the stability of steadystate motions of mechanical system. This method enables one to obtain the sufficient, and in some cases the necessary, stability conditions.

The present paper concerns certain aspects and further possibilities of the above method, including its applicability to nonholonomic systems. Its relationship to the Chetaev method for constructing Liapunov functions is considered. The discussion is illustrated with examples.

Let us consider some mechanical system whose phase variables characterizing its position and velocities at any instant t (or some of these variables) are x_s ($s = 1, \ldots, n$). We assume that the variables x_s are independent if the system is holonomic, or that they may be related by some nonintegrable constraining equations if the system is nonholonomic. As these variables we can take, for example, the Lagrange variables of the system q_j, q_j; other possibilities are to take certain nonholonomic coordinates or quasi-coordinates.

Let us assume that some number of independent first integrals

$$F_i, (x_1, \ldots, x_n) = c_i \qquad (i = 1, \ldots, m, \quad m < n) \tag{1}$$

not explicitly dependent on time are known for the differential equations of motion of the system written in one way or another; c_i are arbitrary integration constants.

Let us recall the theorem of Routh [2] with Liapunov's important addendum [3].

Theorem. If some number of integrals not explicitly dependent on time has been obtained for the differential equations of motion of some system, and if among these integrals there is one which has a minimum or a maximum for all the given values of the remaining integrals as well as for all of their values which are sufficiently close to the given ones, and, finally, if the values of the variables in the integral which deliver its extremum are continuous functions of the values of these integrals, then the motion of the system for certain values of the variables which minimize or maximize the integral in question for the given values of the other integrals is stable with respect to these variables for all sufficiently small perturbations.

Liapunov did not prove this theorem, apparently regarding it as self-evident. It is possible, in fact, to adduce a very simple proof [4], whose idea can be stated briefly as follows.

Let $F_1(x_1, \ldots, x_n) = c_1$ be the integral referred to in the theorem.

Since, by hypothesis, this integral has a minimum or maximum both for given values of the constants $c_j = c_j^\circ$ and for all sufficiently close values $c_j = c_j^\circ + \Delta c_j$ ($j = 2, \ldots, m$) of